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ASYMPTOTIC INTEGRATION OF THE DYNAMIC EQUATIONS OF THE THEORY OF ELASTICITY FOR THE CASE OF THIN SHELLS†

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Asymptotic integration of the three-dimensional dynamic equations of the theory of elasticity is carried out for the case of thin shells. Specific features of the asymptotic properties of the stress–strain state (SSS) of the shell in dynamic problems are discussed. Limiting two-dimensional systems of equations are derived.

ALTHOUGH there are many publications ([1–3], and others) devoted to the application of asymptotic methods to problems concerned with the dynamics of shells in the two-dimensional setting, no asymptotic derivation of the two-dimensional dynamic equations of shell theory from the three-dimensional equations of the theory of elasticity has been carried out. Only the integrals of the dynamic equations of the theory of elasticity for the case of thin shells have been considered [4, 5].

Since there is a significant difference between the asymptotic properties of the stress–strain state (SSS) of a shell in dynamics and statics, the dynamic case calls for a special study and cannot be reduced to a formal incorporation of inertial terms. Besides, the study of the asymptotic behaviour of the parameters of the SSS across the shell is important in dynamic problems in which it is required to prove the existence of domains (or intervals) in which the solutions obtained by means of the two-dimensional shell theory and the boundary-layer theory [3, 4] agree with one another.

1. EQUATIONS OF THE THREE-DIMENSIONAL THEORY OF ELASTICITY

We consider a thin elastic shell with relative half-thickness $\eta = h/R$ ($2h$ is the thickness of the shell and R is the characteristic radius of curvature of the middle surface of the shell).

We take the dynamic equations of the theory of elasticity describing the motion of the shell as a three-dimensional elastic body in the form [6]

$$L_i + \left(1+\eta\frac{\zeta}{R_i^*}\right)^{-1}\frac{1}{R\eta}\frac{\partial}{\partial\zeta}\left[\left(1+\eta\frac{\zeta}{R_i^*}\right)^2\tau_{i3}\right] - \left(1+\eta\frac{\zeta}{R_i^*}\right)\left(1+\eta\frac{\zeta}{R_j}\right)\rho\left(\frac{c_s\eta^{-a}}{R}\right)^2\frac{\partial^2 v_i}{\partial\tau^2} =$$

$$= 0$$

$$-L + F + \frac{1}{R\eta}\frac{\partial\tau_3}{\partial\zeta} - \left(1+\eta\frac{\zeta}{R_1}\right)\left(1+\eta\frac{\zeta}{R_2^*}\right)\rho\left(\frac{c_s\eta^{-a}}{R}\right)^2\frac{\partial^2 v_3}{\partial\tau^2} = 0$$

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$$E\left(1+\eta\frac{\zeta}{R_j^*}\right)e_i=\left(1+\eta\frac{\zeta}{R_i^*}\right)\tau_i-\nu\left(1+\eta\frac{\zeta}{R_j^*}\right)\tau_j-\nu\tau_3 \tag{1.1}$$

$$\frac{E}{R\eta}\left(1+\eta\frac{\zeta}{R_1^*}\right)\left(1+\eta\frac{\zeta}{R_2^*}\right)\frac{\partial v_3}{\partial\zeta}=\tau_3-\nu\left(1+\eta\frac{\zeta}{R_1^*}\right)\tau_1-\nu\left(1+\eta\frac{\zeta}{R_2^*}\right)\tau_2$$

$$\frac{E}{R\eta}\left(1+\eta\frac{\zeta}{R_i^*}\right)\left(1+\eta\frac{\zeta}{R_j^*}\right)\frac{\partial v_i}{\partial\zeta}+E\left(1+\eta\frac{\zeta}{R_j^*}\right)g_i=2(1+\nu)\left(1+\eta\frac{\zeta}{R_j^*}\right)\tau_{i3}$$

$$E\left(1+\eta\frac{\zeta}{R_i^*}\right)m_i+E\left(1+\eta\frac{\zeta}{R_j^*}\right)m_j=2(1+\nu)\left(1+\eta\frac{\zeta}{R_j^*}\right)\tau_{ij}$$

$$L_i=\frac{\eta^{-q}}{R}\left[\frac{1}{A_i}\frac{\partial\tau_i}{\partial\xi_i}+\frac{1}{A_j}\frac{\partial\tau_{ij}}{\partial\xi_j}+\frac{\eta^q}{A_iA_j}\frac{\partial A_j}{\partial\xi_{io}}(\tau_i-\tau_j)+\right.$$

$$\left.+\frac{\eta^q}{A_iA_j}\frac{\partial A_i}{\partial\xi_{jo}}(\tau_{ij}+\tau_{ji})\right],\quad L=\frac{1}{R}\left(\frac{\tau_1}{R_1^*}+\frac{\tau_2}{R_2^*}\right),\quad F=\frac{\eta^{-q}}{R}\left(\frac{1}{A_1}\frac{\partial\tau_{13}}{\partial\xi_1}+\right.$$

$$\left.+\frac{1}{A_2}\frac{\partial\tau_{23}}{\partial\xi_2}+\frac{\eta^q}{A_1A_2}\frac{\partial A_2}{\partial\xi_{10}}\tau_{13}+\frac{\eta^q}{A_1A_2}\frac{\partial A_1}{\partial\xi_{20}}\tau_{23}\right) \tag{1.2}$$

$$e_i=\frac{\eta^{-q}}{R}\left(\frac{1}{A_i}\frac{\partial v_i}{\partial\xi_i}+\frac{\eta^q}{A_iA_j}\frac{\partial A_i}{\partial\xi_{jo}}v_j+\eta^q\frac{v_3}{R_i^*}\right)$$

$$m_i=\frac{\eta^{-q}}{R}\left(\frac{1}{A_j}\frac{\partial v_i}{\partial\xi_j}-\frac{\eta^q}{A_iA_j}\frac{\partial A_j}{\partial\xi_{io}}v_j\right),\quad g_i=\frac{\eta^{-q}}{R}\left(\frac{1}{A_i}\frac{\partial v_3}{\partial\xi_i}-\eta^q\frac{v_i}{R_i^*}\right)$$

$$\xi_{io}=\eta^q\xi_i,\quad R_i^*=R_i/R$$

$$\tau_i=\left(1+\eta\frac{\zeta}{R_j^*}\right)\sigma_{ii},\quad \tau_{ij}=\left(1+\eta\frac{\zeta}{R_i^*}\right)\sigma_{ij}$$

$$\tau_{i3}=\tau_{3i}=\left(1+\eta\frac{\zeta}{R_j^*}\right)\sigma_{i3},\quad \tau_3=\left(1+\eta\frac{\zeta}{R_1^*}\right)\left(1+\eta\frac{\zeta}{R_2^*}\right)\sigma_{33}$$

Here $i\neq j=1, 2$, σ_{kl} $(k, l=1, 2, 3)$ are the stresses, v_m $(m=1, 2, 3)$ are the displacements, E is Young's modulus, ν is Poisson's ratio, ρ is the density of the material of the shell, A_k and R_m are the coefficients of the first quadratic form and the principal radii of curvature of the middle surface $\zeta=0$ of the shell, q is the variability index of the SSS of the shell, and a is the dynamics index. The dimensionless variables ξ_i, ζ, τ are related to their dimensional analogues α_k, t (α_i are the parameters of the curvature lines in the middle surface, α_3 is the distance along the normal line to the middle plane, and t is the time) by the following scaling relations

$$\alpha_i=R\eta^q\xi_i,\quad \alpha_3=R\eta\zeta,\quad t=Rc_s^{-1}\eta^a\tau,\quad c_s=\sqrt{E/\rho} \tag{1.3}$$

It is assumed that differentiation with respect to the dimensionless variables does not affect the asymptotic order of the original quantities and the variability and dynamics indices satisfy the inequalities

$$q<1,\quad a<1 \tag{1.4}$$

These inequalities restrict the length of the deformation and the characteristic time scale of the processes being studied and constitute a necessary condition for the application of any two-dimensional shell theory.

We will assume that the front surfaces of the shell are free of any external loads, i.e.

$$\tau_{3i}(\xi_1, \xi_2, \pm 1) = \tau_3(\xi_1, \xi_2, \pm 1) = 0 \tag{1.5}$$

and we will focus our attention on the following two cases, that are most important for applications: $q = a$ and $q = (1+a)/2$ $(\alpha \geqslant 0)$. In what follows it will be demonstrated that the first case encompasses the momentum-free $(a = 0)$ and planar $(a > 0)$ integrals of the dynamic equations of the theory of elasticity, while the other case contains the planar-bending integrals $(a = 0)$ as well as bending integrals $(a > 0)$. For the momentum-free integrals $v_3 \sim v_i$, for the planar integrals $v_i \gg v_3$, and for the bending and planar-bending integrals $v_3 \gg v_i$.

2. ASYMPTOTIC INTEGRATION IN THE CASE WHEN $q = a$

We take the asymptotic forms of the SSS of the shell in the form

$$\begin{aligned} &v_i = R\eta^q(v_i^0 + \eta v_i^1), v_3 = R\eta(v_3^0 + \eta^{2q-1}v_3^1) \\ &\tau_i = E(\tau_i^0 + \eta\tau_i^1), \tau_{ij} = E(\tau_{ij}^0 + \eta\tau_{ij}^1) \\ &\tau_{i3} = E\eta^{3-3q}(\tau_{i3}^0 + \eta^{2q-1}\tau_{i3}^1), \ \tau_3 = E\eta^{2-2q}(\tau_3^0 + \eta\tau_3^1) \end{aligned} \tag{2.1}$$

Here it is assumed that all the quantities with zero and unity superscripts have the same asymptotic order. The quantities with superscript zero define the SSS that is symmetric relative to the middle surface of the shell (τ_i^0, τ_{ij}^0, τ_3^0, v_i^0 are even functions of ζ, and τ_{i3}^0, v_3^0 are odd functions of ζ). The quantities with superscript one define the SSS that is antisymmetric relative to the middle surface (τ_i^1, τ_{ij}^1, τ_3^1, v_i^1 are odd functions of ζ, while τ_{i3}^1, v_3^1 are even functions of ζ).

The method connected with the decomposition of the SSS of the shell into the symmetric and antisymmetric components has been applied before in the case of the asymptotic integration of the dynamic equations of the theory of elasticity in the neighbourhood of the frequencies of the section [7].

For $q = 0$ the asymptotic form (2.1) coincides with that of a static momentum-free SSS [6]. For $q = 1$ the SSS of the shell is symmetric relative to the middle surface apart from the value of $O(\eta)$. With this accuracy the Lamé coefficients of the chosen three-orthogonal coordinate system are constant in ζ.

On substituting (2.1) into (1.2), we distinguish the even functions (L_i^0, L^0, F^1, e_i^0, m_i^0, g_i^1) and odd function (L_i^1, L^1, Γ^0, e_i^1, m_i^1, g_i^0) of the transversal coordinate

$$\begin{aligned} &L_i = \frac{E}{R}\eta^{-q}(L_i^0 + \eta L_i^1), \ L = \frac{E}{R}(L^0 + \eta L^1), F = \frac{E}{R}\eta^{3-4q}(F^0 + \eta^{2q-1}F^1) \\ &e_i = e_i^0 + \eta e_i^1, m_i = m_i^0 + \eta m_i^1, g_i = \eta^{1-q}(g_i^0 + \eta^{2q-1}g_i^1) \\ &L_i^k = \frac{1}{A_i}\frac{\partial\tau_i^k}{\partial\xi_i} + \frac{1}{A_j}\frac{\partial\tau_{ij}^k}{\partial\xi_j} + \frac{\eta^q}{A_iA_j}\frac{\partial A_j}{\partial\xi_{i0}}(\tau_i^k - \tau_j^k) + \\ &+ \frac{\eta^q}{A_iA_j}\frac{\partial A_i}{\partial\xi_{j0}}(\tau_{ij}^k + \tau_{ji}^k), \ L^k = \frac{\tau_1^k}{R_1^*} + \frac{\tau_2^k}{R_2^*} \\ &F^k = \frac{1}{A_1}\frac{\partial\tau_{13}^k}{\partial\xi_1} + \frac{1}{A_2}\frac{\partial\tau_{23}^k}{\partial\xi_2} + \frac{\eta^q}{A_1A_2}\frac{\partial A_2}{\partial\xi_{10}}\tau_{13}^k + \frac{\eta^q}{A_1A_2}\frac{\partial A_1}{\partial\xi_{20}}\tau_{23}^k \end{aligned} \tag{2.2}$$

$$e_i^0 = \frac{1}{A_i}\frac{\partial v_i^0}{\partial \xi_i} + \frac{\eta^q}{A_i A_j}\frac{\partial A_i}{\partial \xi_{j0}} v_j^0 + \eta^{2q}\frac{v_3^1}{R_i^*}$$

$$e_i^1 = \frac{1}{A_i}\frac{\partial v_i^1}{\partial \xi_i} + \frac{\eta^q}{A_i A_j}\frac{\partial A_i}{\partial \xi_{j0}} v_j^1 + \frac{v_3^0}{R_i^*}$$

$$m_i^k = \frac{1}{A_j}\frac{\partial v_i^k}{\partial \xi_j} - \frac{\eta^q}{A_i A_j}\frac{\partial A_j}{\partial \xi_{i0}} v_j^k$$

$$g_i^0 = \frac{1}{A_i}\frac{\partial v_3^0}{\partial \xi_i} - \eta^{2q}\frac{v_i^1}{R_i^*},\quad g_i^1 = \frac{1}{A_i}\frac{\partial v_3^1}{\partial \xi_i} - \frac{v_i^0}{R_i^*}\quad (k = 0, 1)$$

From the form of (2.1) it follows that in the case under consideration the even functions v_i^0, τ_i^0, τ_{ij}^0, as well as v_3^1 for $q = 0$, will be the principal asymptotic parameters of the SSS of the shell. Substituting (2.1) and (2.2) into (1.1) and discarding terms of order $O(\eta^{2-2q})$ as compared to unity, we obtain a closed system of equations for the principal parameters of the SSS. It has the form

$$L_i^0 - \frac{\partial^2 v_i^0}{\partial \tau^2} = 0,\quad L^0 + \frac{\partial^2 v_3^1}{\partial \tau^2} = 0$$
$$e_i^0 = \tau_i^0 - \nu\tau_j^0,\quad \frac{\partial v_3^1}{\partial \zeta} = 0,\quad \frac{\partial v_i^0}{\partial \zeta} = 0,\quad m_i^0 + m_j^0 = 2(1+\nu)\tau_{ij}^0 \tag{2.3}$$

The admissible error $O(\eta^{2-2q})$ is the same as that of the slightly modified Kirchhoff–Love shell theory in statics [6]. All the remaining quantities in (2.1), except for τ_{i3}^0 and τ_3^1, can be determined to within this error from the known variables v_i^0, τ_i^0, τ_{ij}^0 and v_3^1 using the following system of equations

$$L_i^1 + \frac{\partial \tau_{i3}^1}{\partial \zeta} - \frac{\partial^2 v_i^1}{\partial \tau^2} - \zeta\left(\frac{1}{R_i^*} + \frac{1}{R_j^*}\right)\frac{\partial^2 v_i^0}{\partial \tau^2} = 0$$
$$-\eta^{2q}L^1 + \frac{\partial \tau_3^0}{\partial \zeta} - \frac{\partial^2 v_3^0}{\partial \tau^2} - \eta^{2q}\zeta\left(\frac{1}{R_1^*} + \frac{1}{R_2^*}\right)\frac{\partial^2 v_3^1}{\partial \tau^2} = 0$$
$$e_i^1 = \tau_i^1 - \nu\tau_j^1 + \zeta\left(\frac{1}{R_i^*} - \frac{1}{R_j^*}\right)\tau_i^0 \tag{2.4}$$
$$\frac{\partial v_3^0}{\partial \zeta} = -\nu(\tau_1^0 + \tau_2^0),\quad \frac{\partial v_i^1}{\partial \zeta} + g_i^1 = 0$$
$$m_i^1 + m_j^1 = 2(1+\nu)\tau_{ij}^1 - \zeta\left(\frac{1}{R_i^*} - \frac{1}{R_j^*}\right)m_i^0$$

However, the stresses τ_{13}^1, τ_3^0 must satisfy the homogeneous boundary conditions

$$\tau_{i3}^1(\xi_1, \xi_2, \pm 1) = \tau_3^0(\xi_1, \xi_2, \pm 1) = 0 \tag{2.5}$$

on the side surfaces of the shell.

In order to determine the stresses τ_{13}^0, τ_3^1 one must construct the systems of equations (2.3) and (2.4) with higher accuracy than $O(\eta^{2-2q})$. We shall not dwell on this question here.

Taking the integrals with respect to ζ in (2.3) and then (2.4), in view of (2.5) we find that the studied integrals of the three-dimensional dynamic equations of the theory of elasticity with error $O(\eta^{2-2q})$ have the form of polynomials in the first coordinate

$$\begin{aligned}
&v_i^0 = v_{i,0}^0, v_3^1 = v_{3,0}^1, e_i^0 = e_{i,0}^0, m_i^0 = m_{i,0}^0\\
&g_i^1 = g_{i,0}^1, \tau_i^0 = \tau_{i,0}^0, \tau_{ij}^0 = \tau_{ij,0}^0, L_i^0 = L_{i,0}^0\\
&L^0 = L_{,0}^0, v_i^1 = v_{i,1}^1\zeta, v_3^0 = v_{3,1}^0\zeta, e_i^1 = e_{i,1}^1\zeta\\
&m_i^1 = m_{i,1}^1\zeta, \tau_i^1 = \tau_{i,1}^1\zeta, \tau_{ij}^1 = \tau_{ij,1}^1\zeta, L_i^1 = L_{i,1}^1\zeta\\
&L^1 = L_{,1}^1\zeta, \tau_{i3}^1 = \tau_{i3,0}^1 + \tau_{i3,2}^1\zeta^2, \tau_3^0 = \tau_{3,0}^0 + \tau_{3,2}^0\zeta^2
\end{aligned} \tag{2.6}$$

All the functions in (2.6) with a comma in the subscript are independent of the transversal coordinate ζ and are related by the formulae

$$\begin{aligned}
&e_{i,0}^0 = \frac{1}{A_i}\frac{\partial v_{i,0}^0}{\partial \xi_i} + \frac{\eta^q}{A_iA_j}\frac{\partial A_i}{\partial \xi_{i0}} v_{j,0} + \eta^{2q}\frac{v_{3,0}^1}{R_i^*}\\
&m_{i,0}^0 = \frac{1}{A_j}\frac{\partial v_{i,0}^0}{\partial \xi_j} - \frac{\eta^q}{A_iA_j}\frac{\partial A_j}{\partial \xi_{i0}} v_{j,0}^0,\quad g_{i,0}^1 = \frac{1}{A_i}\frac{\partial v_{3,0}^1}{\partial \xi_i} - \frac{v_{i,0}^0}{R_i^*}\\
&L_{i,0}^0 = \frac{1}{A_i}\frac{\partial \tau_{i,0}^0}{\partial \xi_i} + \frac{1}{A_j}\frac{\partial \tau_{ij,0}^0}{\partial \xi_j} + \frac{\eta^q}{A_iA_j}\frac{\partial A_j}{\partial \xi_{i0}}(\tau_{i,0}^0 - \tau_{j,0}^0) +\\
&+ \frac{\eta^q}{A_iA_j}\frac{\partial A_i}{\partial \xi_{j0}}(\tau_{ij,0}^0 + \tau_{ji,0}^0),\quad L_{,0}^0 = \frac{\tau_{1,0}^0}{R_1^*} + \frac{\tau_{2,0}^0}{R_2^*}\\
&\tau_{i,0}^0 = \frac{1}{1-\nu^2}(e_{i,0}^0 + \nu e_{j,0}^0),\quad \tau_{ij,0}^0 = \frac{1}{2(1+\nu)}(m_{i,0}^0 + m_{j,0}^0)\\
&L_{i,0}^0 - \frac{\partial^2 v_{i,0}^0}{\partial \tau^2} = 0,\quad L_{,0}^0 + \frac{\partial^2 v_{3,0}^1}{\partial \tau^2} = 0\\
&v_{i,1}^1 = -g_{i,0}^1,\quad v_{3,1}^0 = -\nu(\tau_{1,0}^0 + \tau_{2,0}^0)\\
&e_{i,1}^1 = \frac{1}{A_i}\frac{\partial v_{i,1}^1}{\partial \xi_i} + \frac{\eta^q}{A_iA_j}\frac{\partial A_i}{\partial \xi_{j0}} v_{j,1}^1 + \frac{v_{3,1}^0}{R_i^*}\\
&m_{i,1}^1 = \frac{1}{A_j}\frac{\partial v_{i,1}^1}{\partial \xi_j} - \frac{\eta^q}{A_iA_j}\frac{\partial A_j}{\partial \xi_{i0}} v_{j,1}^1\\
&\tau_{i,1}^1 = \frac{1}{1-\nu^2}(e_{i,1}^1 + \nu e_{j,1}^1) + \frac{1}{1-\nu^2}\left(\frac{1}{R_j^*} - \frac{1}{R_i^*}\right)(\tau_{i,0}^0 - \nu\tau_{j,0}^0)\\
&\tau_{ij,1}^1 = \frac{1}{2(1+\nu)}(m_{i,1}^1 + m_{j,1}^1)\\
&L_{i,1}^1 = \frac{1}{A_i}\frac{\partial \tau_{i,1}^1}{\partial \xi_i} + \frac{1}{A_j}\frac{\partial \tau_{ij,1}^1}{\partial \xi_j} + \frac{\eta^q}{A_iA_j}\frac{\partial A_j}{\partial \xi_{i0}}(\tau_{i,1}^1 - \tau_{j,1}^1) +\\
&+ \frac{\eta^q}{A_iA_j}\frac{\partial A_i}{\partial \xi_{j0}}(\tau_{ij,1}^1 + \tau_{ji,1}^1),\quad L_{,1}^1 = \frac{\tau_{1,1}^1}{R_1^*} + \frac{\tau_{2,1}^1}{R_2^*}\\
&\tau_{i3,0}^1 = -\tau_{i3,2}^1,\quad \tau_{i3,2}^1 = -\frac{1}{2}\left[L_{i,1}^1 + \frac{\partial^2 g_{i,0}^1}{\partial \tau^2} - \left(\frac{1}{R_i^*} + \frac{1}{R_j^*}\right)\frac{\partial^2 v_{i,0}^0}{\partial \tau^2}\right]
\end{aligned} \tag{2.7}$$

$$\tau_{3,0}^0 = -\tau_{3,2}^0,\ \tau_{3,2}^0 = \frac{1}{2}\left[\frac{\partial^2 v_{3,1}^0}{\partial \tau^2} + \eta^{2q} L_{,1}^1 + \eta^{2q}\left(\frac{1}{R_1^*} + \frac{1}{R_2^*}\right)\frac{\partial^2 v_{3,1}^1}{\partial \tau^2}\right]$$

To justify the asymptotic integration performed above, one must substitute relations (2.1) and (2.2) into (1.1) and (1.2) after expressing all the quantities in these relations by (2.6) and (2.7). By means of identity transformations it can be verified directly that the resulting discrepancy will be of order η^{2-2q}.

3. ASYMPTOTIC INTEGRATION IN THE CASE WHEN $q=\frac{1}{2}(1+a)$

We take the asymptotic form of the SSS of the shell to be

$$\begin{aligned} &v_i = R\eta(\eta^{2q-1} v_i^0 + v_i^1),\ v_3 = R\eta^q(\eta v_3^0 + v_3^1)\\ &\tau_i = E\eta^{1-q}(\eta^{2q-1}\tau_i^0 + \tau_i^1),\ \tau_{ij} = E\eta^{1-q}(\eta^{2q-1}\tau_{ij}^0 + \tau_{ij}^1)\\ &\tau_{i3} = E\eta^{2-2q}(\eta\,\tau_{i3}^0 + \tau_{i3}^1),\ \tau_3 = E\eta^{3-3q}(\eta^{2q-1}\tau_3^0 + \tau_3^1)\end{aligned} \tag{3.1}$$

The quantities with zero and unity superscripts have the same meaning as in Sec. 2.

Substituting (3.1) into (1.2), we separate the even and odd functions of the transversal coordinate by means of the formulae

$$\begin{aligned} &L_i = \frac{E}{R}\eta^{1-2q}(\eta^{2q-1}L_i^0 + L_i^1),\ L = \frac{E}{R}\eta^{1-q}(\eta^{2q-1}L^0 + L^1)\\ &F = \frac{E}{R}\eta^{2-3q}(\eta F^0 + F^1)\\ &e_i = \eta^{1-q}(\eta^{2q-1}e_i^0 + e_i^1),\ m_i = \eta^{1-q}(\eta^{2q-1}m_i^0 + m_i^1),\ g_i = \eta g_i^0 + g_i^1\end{aligned} \tag{3.2}$$

where L_i^k, L^k, F^k, m_i^k are defined by (2.2) as before, and where the following expressions are satisfied for the remaining quantities

$$\begin{aligned} &e_i^0 = \frac{1}{A_i}\frac{\partial v_i^0}{\partial \xi_i} + \frac{\eta^q}{A_iA_j}\frac{\partial A_i}{\partial \xi_{j0}} v_j^0 + \frac{v_3^1}{R_i^*}\\ &e_i^1 = \frac{1}{A_i}\frac{\partial v_i^1}{\partial \xi_i} + \frac{\eta^q}{A_iA_j}\frac{\partial A_i}{\partial \xi_{j0}} v_j^1 + \eta^{2q}\frac{v_3^0}{R_i^*}\\ &g_i^0 = \frac{1}{A_i}\frac{\partial v_3^0}{\partial \xi_i} - \frac{v_i^1}{R_i^*},\\ &g_i^1 = \frac{1}{A_i}\frac{\partial v_3^1}{\partial \xi_i} - \eta^{2q}\frac{v_i^0}{R_i^*}\end{aligned} \tag{3.3}$$

Substituting (3.1) and (3.2) into (1.1) and discarding all terms of order $O(\eta^{2-2q})$ we arrive at equations of the form

$$L_i^0 = 0,\ L_i^1 + \frac{\partial \tau_{i3}^1}{\partial \zeta} = 0,\ -\eta^{4q-2}L^0 + F^1 + \frac{\partial \tau_3^1}{\partial \zeta} - \frac{\partial^2 v_3^1}{\partial \tau^2} = 0$$

$$-L^1 + \frac{\partial \tau_3^0}{\partial \zeta} = 0,\ e_i^0 = \tau_i^0 - \nu\tau_j^0,\ e_i^1 = \tau_i^1 - \nu\tau_j^1,\ \frac{\partial v_3^1}{\partial \zeta} = 0$$

$$\frac{\partial v_3^0}{\partial \zeta} = -\nu(\tau_1^0 + \tau_2^0), \frac{\partial v_i^0}{\partial \zeta} = 0, \frac{\partial v_i^1}{\partial \zeta} + g_i^1 = 0 \tag{3.4}$$

$$m_i^0 + m_j^0 = 2(1+\nu)\tau_{ij}^0, m_i^1 + m_j^1 = 2(1+\nu)\tau_{ij}^1$$

The stresses τ_{i3}^1, τ_3^0, τ_3^1 must satisfy the homogeneous boundary conditions on the side surfaces $\zeta = \pm 1$ of the shell. In order to determine τ_{i3}^0 it is necessary to set up (3.4) with accuracy exceeding $O(\eta^{2-2q})$.

Under the given boundary conditions on the side surfaces of the shell, the required integrals of the system of equations (3.4) are determined apart to the value of $O(\eta^{2-2q})$ by (2.6) and the formulae

$$F^1 = F_0^1 + F_2^1 \zeta^2, \ \tau_3^1 = \tau_{3,1}^1 \zeta + \tau_{3,3}^1 \zeta^3 \tag{3.5}$$

All two-dimensional quantities in (2.6) and (3.5) are related by formulae (2.7) in which the expressions for $e_{i,0}^0$, $e_{i,1}^1$, $g_{i,0}^1$, $\tau_{i,1}^1$, $\tau_{i2,2}^1$, $\tau_{i3,2}^0$ must be changed and expressions for $\tau_{3,1}^1$, $\tau_{3,3}^1$ must be added. The expressions have the form

$$\begin{aligned}
e_{i,0}^0 &= \frac{1}{A_i}\frac{\partial v_{i,0}^0}{\partial \xi_i} + \frac{\eta^q}{A_i A_j}\frac{\partial A_i}{\partial \xi_{j0}} v_{j,0}^0 + \frac{v_{3,0}^1}{R_i^*} \\
e_{i,1}^1 &= \frac{1}{A_i}\frac{\partial v_{i,1}^1}{\partial \xi_i} + \frac{\eta^q}{A_i A_j}\frac{\partial A_i}{\partial \xi_{j0}} v_{j,1}^1, \ g_{i,0}^1 = \frac{1}{A_i}\frac{\partial v_{3,0}^1}{\partial \xi_i} \\
\tau_{i,1}^1 &= \frac{1}{1-\nu^2}(e_{i,1}^1 + \nu e_{j,1}^1), \ \tau_{i3,2}^1 = -\frac{1}{2} L_{i,1}^1, \ \tau_{3,2}^0 = \frac{1}{2} L_{,1}^1 \\
\tau_{3,1}^1 &= \eta^{4q-2} L_{,0}^0 - F_{,0}^1 + \frac{\partial^2 v_{3,0}^1}{\partial \tau^2}, \ \tau_{3,3}^1 = -\frac{1}{3} F_{,2}^1 = -\tau_{3,1}^1, F_{,l}^1 = \\
&= \frac{1}{A_1}\frac{\partial \tau_{13,l}^1}{\partial \xi_1} + \frac{1}{A_2}\frac{\partial \tau_{23,l}^1}{\partial \xi_2} + \frac{\eta^q}{A_1 A_2}\frac{\partial A_2}{\partial \xi_{10}} \tau_{13,l}^1 + \frac{\eta^q}{A_1 A_2}\frac{\partial A_1}{\partial \xi_{20}} \tau_{23,l}^1, \ l = 0,2
\end{aligned} \tag{3.6}$$

respectively.

Reasoning similar to that in Sec. 2 makes it possible to verify that the constructed integrals of the three-dimensional dynamic equations of the theory of elasticity have all the *a priori* assumed asymptotic properties.

4. TWO-DIMENSIONAL EQUATIONS INVOLVING FORCES AND MOMENTA

Following the traditional approach, we will obtain the equations of motion for the shell in terms of the forces and momenta. We begin with the case when $q = a$. We will derive a two-dimensional system of equations for determining the asymptotically principal parameters of the SSS of the shell. We introduce the notation

$$\begin{aligned}
&T_i = 2Eh\tau_{i,0}^0, \ S_{ij} = 2Eh\tau_{ij,0}^0, \ u_i = R\eta^q v_{i,0}^0 \\
&w = -R\eta^{2q} v_{3,0}^1, \ \epsilon_i = e_{i,0}^0, \ \omega = m_{i,0}^0 + m_{j,0}^0
\end{aligned} \tag{4.1}$$

Here T_i and S_{ij} are the normal and tangential stresses, u_i are the tangential displacements of the middle surface, w is the deflection of the middle surface, and ϵ_i and ω are the components of the tangential deformation.

Substituting (4.1) into (2.7) and taking (1.3) into account, we get

$$\frac{1}{A_i}\frac{\partial T_i}{\partial \alpha_i}+\frac{1}{A_j}\frac{\partial S_{ij}}{\partial \alpha_j}+\frac{1}{A_iA_j}\frac{\partial A_j}{\partial \alpha_i}(T_i-T_j)+\frac{1}{A_iA_j}\frac{\partial A_i}{\partial \alpha_j}(S_{ij}+S_{ji})-$$
$$-2\rho h\frac{\partial^2 u_i}{\partial t^2}=0,\ \frac{T_1}{R_1}+\frac{T_2}{R_2}-2\rho h\frac{\partial^2 w}{\partial t^2}=0$$
$$T_i=\frac{2Eh}{1-\nu^2}(\epsilon_i+\nu\epsilon_j),\ S_{ij}=\frac{Eh}{1+\nu}\omega,\ \epsilon_i=\frac{1}{A_i}\frac{\partial u_i}{\partial \alpha_i}+ \tag{4.2}$$
$$+\frac{1}{A_iA_j}\frac{\partial A_i}{\partial \alpha_j}u_j-\frac{w}{R_i},\ \omega=\frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{u_1}{A_1}\right)+\frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{u_2}{A_2}\right)$$

If $q=a=0$, then formulae (4.1) define the dynamic momentum-free integrals of the system of equations (4.2). For these integrals $v_i \sim v_3$. For $q=a>0$ the integrals (4.2) are planar ($v_i \gg v_3$). In this case, with accuracy $O(\eta^{2q})$ as compared to unity, one can neglect the term w/R_i in the expression for ϵ_i (see (4.1) and (1.3)). To within this accuracy the first and third through to the sixth equations in (4.2) can be taken to form a separate subsystem, which, to within the metric, coincides with the equations of the plane problem of the size of elasticity for the case of a generalized plane stress state. In statics such a situation can be realized only if $q>\frac{1}{2}$ [6]. This is the main difference between the static and dynamic cases.

The asymptotic behaviour of the two-dimensional quantities (4.1) and the corresponding asymptotic behaviour of the three-dimensional quantities (2.1) is specific of dynamic cases only. Physically it means that in dynamic cases the projection of the tangential forces onto the normal direction to the middle surface can be compensated by the transverse inertia of the shell.

Starting from (2.7) one can also obtain the two-dimensional equations for those parameters of the SSS of the shell that are asymptotically of the higher order. Without dwelling on this point, we shall only state the expression for the transverse compression $m=hv_{3,1}^0$ of the shell, which exceeds asymptotically the deflection w if $q>\frac{1}{2}$. It has the form

$$m=-\frac{\nu}{2E}(T_1+T_2) \tag{4.3}$$

In the case when $q=\frac{1}{2}(1+a)$ we introduce the notation

$$T_i=2Eh\eta^q\tau_{i,0}^0,\ S_{ij}=2Eh\eta^q\tau_{ij,0}^0,\ G_i=-\tfrac{2}{3}Eh^2\eta^{1-q}\tau_{ij,1}^1$$
$$H_{ij}=\tfrac{2}{3}Eh^2\eta^{1-q}\tau_{ij,1}^1,\ N_i=-2Eh\eta^{2-2q}(\tau_{i3,0}^1+\tfrac{1}{3}\tau_{i3,2}^1)$$
$$u_i=R\eta^{2q}v_{i,0}^0,\ w=-R\eta^q v_{3,0}^1,\ \gamma_i=-v_{i,1}^1 \tag{4.4}$$
$$\kappa_i=R^{-1}\eta^{-q}e_{i,1}^1,\ \tau=R^{-1}\eta^{-q}m_{i,1}^1$$

Here G_i and H_{ij} are the bending momenta and torques, N_i are the cutting forces, γ_i are the twists, and κ_i and τ are the components of the bending deformation of the middle plane. The remaining quantities have the same meaning as above.

As opposed to (4.1), the asymptotic form (4.4) remains valid in statics. It corresponds to integrals for which $v_3 \gg v_i$.

Substituting (4.4) into (2.7) and (3.6) and taking (1.3) into account, we obtain a system of equations of the form

$$\frac{1}{A_i}\frac{\partial T_i}{\partial \alpha_i}+\frac{1}{A_j}\frac{\partial S_{ij}}{\partial \alpha_j}+\frac{1}{A_iA_j}\frac{\partial A_j}{\partial \alpha_i}(T_i-T_j)+\frac{1}{A_iA_j}\frac{\partial A_i}{\partial \alpha_j}(S_{ij}+S_{ji})=0$$

$$\frac{T_1}{R_1}+\frac{T_2}{R_2}+\frac{1}{A_1}\frac{\partial N_1}{\partial \alpha_1}+\frac{1}{A_2}\frac{\partial N_2}{\partial \alpha_2}+\frac{1}{A_1A_2}\frac{\partial A_2}{\partial \alpha_1}N_1+\frac{1}{A_1A_2}\frac{\partial A_1}{\partial \alpha_2}N_2-2\rho h\frac{\partial^2 w}{\partial t^2}=0$$

$$\frac{1}{A_i}\frac{\partial G_i}{\partial \alpha_i}-\frac{1}{A_j}\frac{\partial H_{ij}}{\partial \alpha_j}+\frac{1}{A_iA_j}\frac{\partial A_j}{\partial \alpha_i}(G_i-G_j)-\frac{1}{A_iA_j}\frac{\partial A_i}{\partial \alpha_j}(H_{ij}+H_{ji})-N_i=0$$

$$T_i=\frac{2Eh}{1-\nu^2}(\epsilon_i+\nu\epsilon_j),\ S_{ij}=\frac{Eh}{1+\nu}\omega$$

$$G_i=-\frac{2Eh^3}{3(1-\nu^2)}(\kappa_i+\nu\kappa_j),\ H_{ij}=\frac{2Eh^3}{3(1+\nu)}\tau \tag{4.5}$$

$$\epsilon_i=\frac{1}{A_i}\frac{\partial u_i}{\partial \alpha_i}+\frac{1}{A_iA_j}\frac{\partial A_i}{\partial \alpha_j}u_j-\frac{w}{R_i},\ \omega=\frac{A_1}{A_2}\frac{\partial}{\partial \alpha_2}\left(\frac{u_1}{A_1}\right)+\frac{A_2}{A_1}\frac{\partial}{\partial \alpha_1}\left(\frac{u_2}{A_2}\right)$$

$$\kappa_i=-\frac{1}{A_i}\frac{\partial \gamma_i}{\partial \alpha_i}-\frac{1}{A_iA_j}\frac{\partial A_i}{\partial \alpha_j}\gamma_j$$

$$\gamma_i=-\frac{1}{A_i}\frac{\partial w}{\partial \alpha_i},\ \tau=-\frac{1}{A_j}\frac{\partial \gamma_i}{\partial \alpha_j}+\frac{1}{A_iA_j}\frac{\partial A_j}{\partial \alpha_i}\gamma_j$$

For $q=\frac{1}{2}$ $(a=0)$ Eqs (4.5) describe the so-called planar-bending integrals. They have the same asymptotic order of stresses caused by the tangential forces and momenta (see (3.1) and (4.4)). For $a>0$ the planar-bending integrals turn into bending integrals, for which the stresses due to the momenta exceed asymptotically the stresses caused by the forces. In this case the planar terms (the first two terms in the second equation in (4.5)) are asymptotically of higher order and this equation together with the 3rd, 6th, 7th, 10th, and 11th equations in (4.5) essentially coincide with the bending equations for a plate in the metric of the middle surface of the shell. However, it was demonstrated in [4] that, generally speaking, the above-mentioned planar terms in the second equation in (4.5) must be preserved in the entire domain $0\leq a<1$ because of the required accuracy of the resulting solution.

To conclude, we remark that by retaining simultaneously all the terms appearing in (4.2) and (4.5), we arrive at the complete system of two-dimensional equations of the Kirchhoff–Love dynamic shell theory.

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